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A MATHEMATICAL MODIFICATION OF TWO SURFACE MODEL FORMULATION IN PLASTICITY

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Abstract—Various constitutive models for the description of the elastoplastic deformation with an anisotropic hardening and also a transition from the elastic to the normal-yield (fully-plastic) state have been proposed in the past. Among them the two surface model has been widely used for predictions of metal or soil deformation behavior, which is regarded as a simplification of the multi surface model (a field of hardening moduli) extended from the kinematic hardening model. The past formulation of this model is not given in a mathematically exact form with generality applicable to materials with not only hardening but also softening behavior. In this paper, a reasonable formulation of the two surface model is given by deriving a mathematical condition which must be satisfied in order that surfaces do not intersect in their relative translation and which will be called a "non-intersection condition" and by proposing a reasonable measure to represent an approaching degree to the normal-yield state. This model is extended to the three surface model which enables a smooth elastic–plastic transition to be described. Besides, a loading criterion for hardening/softening materials is derived in a stress formulation. Finally, some comments on past models or formulations are made compared with the present formulation.

INTRODUCTION

An extension of the kinematic hardening model (Edelman and Drucker, 1951; Ishlinski, 1954; Prager, 1956) so as to describe even a deformation proceeding in the transition from the elastic to the normal-yield (fully-plastic) state, which would obey Masing's rule (Masing, 1926), has been attempted by Mroz (1967, 1969) and Iwan (1967) independently. Their extended models are, however, of complex form assuming multiple "subyield (nesting yield) surfaces" encircled by a "normal-yield (bounding or limiting) surface", which has been called a "multi surface model". Thereafter, based on them, a simplified model employing a normal-yield surface and only one "subyield (inner yield) surface" enclosing a purely elastic domain has been formulated by many workers (Dafalias and Popov, 1975, 1976, 1977; Krieg, 1975; Mroz *et al.*, 1979; Hashiguchi, 1981) which has been called a "two surface model".

The past constitutive equations in the framework of the multi or two surface model have not been formulated exactly on the basis of mathematical verifications, however. Especially, little consideration has been given to formulating the mathematical condition which regulates a surface so as not to protrude from an outer larger surface, i.e. which keeps surfaces from intersection at their relative translation. Therefore, even though the past equations could analyze the deformation of specialized materials exhibiting only a hardening behavior, they give rise to a mathematical contradiction on the translation of assumed surfaces in the deformation analysis of generalized material with not only hardening but also softening behavior.

In this paper, the two surface model is formulated in mathematically exact forms. On this formulation the "non-intersection condition" of surfaces is derived, and then the translation rule of the subyield surface is formulated so as to satisfy this condition. Besides, assuming a reasonable measure to describe an approaching degree to the normal-yield state, the plastic strain rate is formulated so as not to violate the non-intersection condition. Further, this model is extended so as to describe the smooth elastic–plastic transition by incorporating the third surface, named a "subloading surface", which exists within the subyield surface and is geometrically similar to it. This model is called a "three surface model", which is regarded as a combination of the two surface model and the subloading surface model (Hashiguchi, 1980). Besides, the loading criterion for hardening/softening materials is derived within the framework of the stress space formulation.

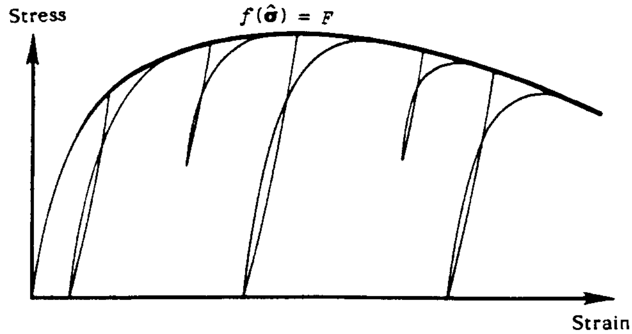

Fig. 1. The normal-yield state illustrated as an envelope curve of reloading state.

BASIC CONSTITUTIVE EQUATIONS IN THE NORMAL-YIELD STATE

A typical stress/strain curve of elastoplastic materials is schematically illustrated in Fig. 1. First, one assumes that the normal-yield state shown by the envelope curve of reloading curves is described by

$$f(\hat{\sigma}) - F(K) = 0 \tag{1}$$

where one sets

$$\hat{\sigma} \equiv \sigma - \hat{\alpha}. \tag{2}$$

The second-order tensor σ is a stress, and the scalar K and the second-order tensor $\hat{\alpha}$ are parameters to describe, respectively, the expansion/contraction and the translation of the surface. For simplicity, one assumes that the surface described by eqn (1), called a "normal-yield surface", expands/contracts retaining a geometrical similarity in a stress space. Therefore, function f is to be a homogeneous function of arguments. Then, let the degree of f be denoted by n.

Let $\dot{K}$ where a superposed dot designates a material time derivative be a function of some plastic internal state variables and the plastic strain rate $\dot{\varepsilon}^p$ in degree one by the dimensional invariance of time.

Further, let $\dot{\hat{\alpha}}$ be given as

$$\dot{\hat{\alpha}} = A\dot{\varepsilon}_v^p \mathbf{1} + B \operatorname{tr}\left(\dot{\varepsilon}^p \frac{\hat{\sigma}}{\|\hat{\sigma}\|}\right) \frac{\hat{\sigma}}{\|\hat{\sigma}\|} \tag{3}$$

in accordance with the previous paper (Hashiguchi, 1981), where

$$\dot{\varepsilon}_v^p \equiv \operatorname{tr} \dot{\varepsilon}^p \tag{4}$$

A ($\geqslant 0$) and B ($\geqslant 0$) are scalar functions of K and $\hat{\alpha}$, and the notation $\|\ \|$ is used to represent the magnitude.

By differentiating eqn (1) and substituting the relation

$$\frac{\partial f}{\partial \hat{\sigma}} = \frac{nF}{\operatorname{tr}(\hat{n}\hat{\sigma})} \hat{n} \tag{5}$$

$$\hat{n} \equiv \frac{\partial f}{\partial \hat{\sigma}} \Big/ \left\| \frac{\partial f}{\partial \hat{\sigma}} \right\| \tag{6}$$

one has the consistency condition

$$\operatorname{tr}\left\{\hat{\boldsymbol{n}}\left(\dot{\boldsymbol{\sigma}}-\frac{1}{n}\frac{\dot{F}}{F}\boldsymbol{\sigma}\right)\right\}=0 \tag{7}$$

which is given as

$$\operatorname{tr}\left[\hat{\boldsymbol{n}}\left\{\dot{\boldsymbol{\sigma}}-A\dot{\varepsilon}^{\mathrm{p}}_{\mathrm{v}}\mathbf{1}-B\operatorname{tr}\left(\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}\frac{\hat{\boldsymbol{\sigma}}}{\|\hat{\boldsymbol{\sigma}}\|}\right)\frac{\hat{\boldsymbol{\sigma}}}{\|\hat{\boldsymbol{\sigma}}\|}-\frac{1}{n}\frac{\dot{F}}{F}\boldsymbol{\sigma}\right\}\right]=0 \tag{8}$$

by incorporating eqn (3).

Here, one assumes that the elastic deformation only can proceed when the stress exists within the normal-yield surface and that the elastic property is not affected by the plastic deformation. Then, in accordance with Drucker's interpretation (Drucker, 1951), one incorporates the associated flow rule

$$\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}=\dot{\lambda}\hat{\boldsymbol{n}} \tag{9}$$

where $\dot{\lambda}$ is a positive proportionality factor. By substituting eqn (9) into eqn (8) the plastic strain rate is given as

$$\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}=\frac{\operatorname{tr}(\hat{\boldsymbol{n}}\dot{\boldsymbol{\sigma}})}{\hat{D}}\hat{\boldsymbol{n}} \tag{10}$$

where

$$\hat{D}\equiv\frac{1}{n}\frac{F'}{F}\hat{\kappa}\operatorname{tr}(\hat{\boldsymbol{n}}\boldsymbol{\sigma})+A\operatorname{tr}^2\hat{\boldsymbol{n}}+B\operatorname{tr}^2\left(\hat{\boldsymbol{n}}\frac{\hat{\boldsymbol{\sigma}}}{\|\hat{\boldsymbol{\sigma}}\|}\right) \tag{11}$$

$$F'\equiv\frac{\mathrm{d}F}{\mathrm{d}K} \tag{12}$$

$\hat{\kappa}$ is a scalar function of stress, plastic internal state variables and $\hat{\boldsymbol{n}}$ in degree one, which is given by

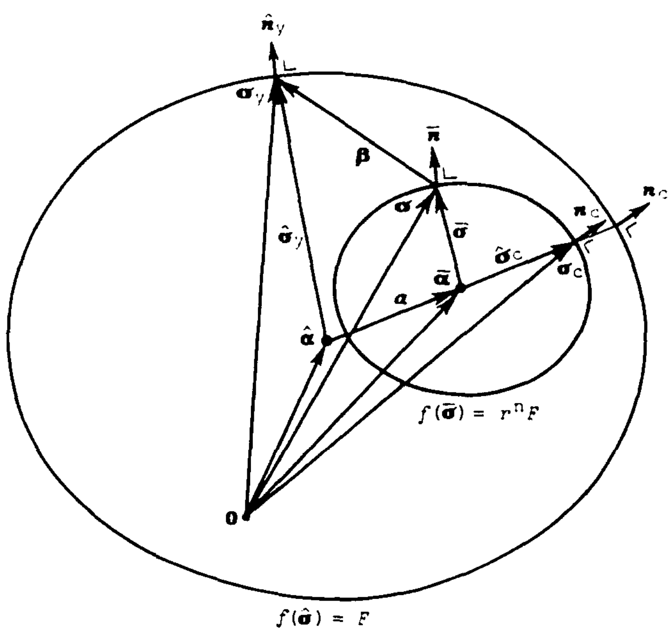

Fig. 2. Normal-yield and subyield surfaces.

$$\dot{\kappa} = K'\dot{\lambda}. \tag{13}$$

TWO SURFACE MODEL

One introduces the subyield surface (Fig. 2) which is geometrically similar to the normal-yield surface and translates within the normal-yield surface. Here, assume that the current stress exists on or within the subyield surface and that the elastoplastic deformation can proceed when it exists on the subyield surface, but only the elastic deformation can proceed when it exists within that surface. Then, let the subyield surface be described by

$$f(\bar{\boldsymbol{\sigma}}) - r^n F(K) = 0 \tag{14}$$

where one sets

$$\bar{\boldsymbol{\sigma}} \equiv \boldsymbol{\sigma} - \bar{\boldsymbol{\alpha}} \tag{15}$$

r $(0 < r < 1)$ is a material constant and the second-order tensor $\bar{\boldsymbol{\alpha}}$ is a parameter to describe a translation of the surface.

Now, let the conjugate point on the normal-yield surface having the same outer-normal direction as that of the subyield surface at the current stress $\boldsymbol{\sigma}$ be denoted by $\boldsymbol{\sigma}_y$ (Fig. 2). Namely, it holds that

$$\hat{\boldsymbol{\sigma}}_y = \frac{1}{r}\bar{\boldsymbol{\sigma}} \tag{16}$$

$$\hat{\boldsymbol{n}}_y = \bar{\boldsymbol{n}} \tag{17}$$

where

$$\hat{\boldsymbol{\sigma}}_y \equiv \boldsymbol{\sigma}_y - \hat{\boldsymbol{\alpha}} \tag{18}$$

$$\hat{\boldsymbol{n}}_y \equiv \frac{\partial f}{\partial \hat{\boldsymbol{\sigma}}_y} \bigg/ \left\| \frac{\partial f}{\partial \hat{\boldsymbol{\sigma}}_y} \right\| \tag{19}$$

$$\bar{\boldsymbol{n}} \equiv \frac{\partial f}{\partial \bar{\boldsymbol{\sigma}}} \bigg/ \left\| \frac{\partial f}{\partial \bar{\boldsymbol{\sigma}}} \right\|. \tag{20}$$

Here, assume that translation rule (3) of the normal-yield surface holds also in the subyield state, provided that the stress in eqn (3) is regarded as the conjugate stress $\boldsymbol{\sigma}_y$. Hence, noting relation (16), one has

$$\dot{\hat{\boldsymbol{\alpha}}} = A\dot{\varepsilon}^p_v \mathbf{1} + B \operatorname{tr}\left(\dot{\boldsymbol{\varepsilon}}^p \frac{\bar{\boldsymbol{\sigma}}}{\|\bar{\boldsymbol{\sigma}}\|}\right) \frac{\bar{\boldsymbol{\sigma}}}{\|\bar{\boldsymbol{\sigma}}\|}. \tag{21}$$

Now, one considers the translation rule of the subyield surface, i.e. $\dot{\bar{\boldsymbol{\alpha}}}$. Since the subyield surface must exist within the normal-yield surface, it must hold that

$$f(\hat{\boldsymbol{\sigma}}_c) \leqslant F \tag{22}$$

where

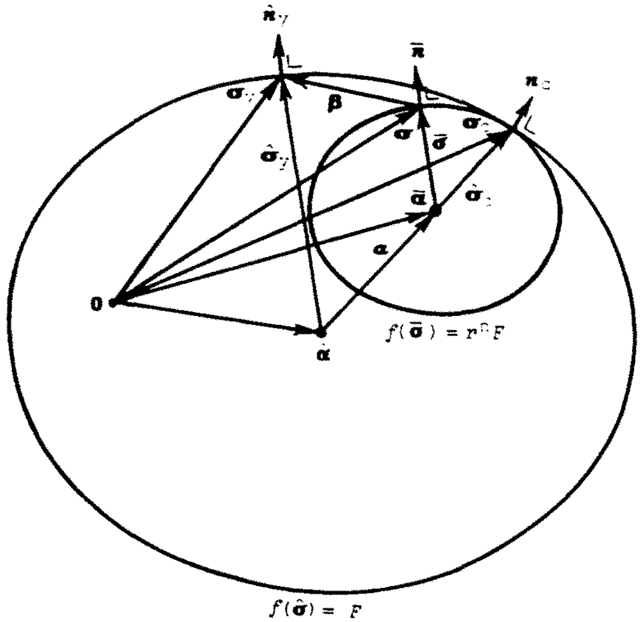

Fig. 3. Normal-yield and subyield surfaces in contact.

$$\hat{\boldsymbol{\sigma}}_c \equiv \boldsymbol{\sigma}_c - \hat{\boldsymbol{\alpha}} \tag{23}$$

letting $\boldsymbol{\sigma}_c$ denote the intersecting point of the subyield surface and the half line starting from the point $\hat{\boldsymbol{\alpha}}$ and passing through the point $\bar{\boldsymbol{\alpha}}$ in the stress space (Fig. 2).

Expression (22) can be written as

$$\operatorname{tr}\left(\frac{\partial f}{\partial \hat{\boldsymbol{\sigma}}_c}\dot{\hat{\boldsymbol{\sigma}}}_c\right) \leqslant \dot{F} \quad \text{when} \quad f(\hat{\boldsymbol{\sigma}}_c) = F \tag{24}$$

in a differential form. Let the condition which must be satisfied in order that the surfaces do not intersect but may contact, such as expression (22) or (24), be called a "non-intersection condition".

In the state that the subyield surface contacts with the normal-yield surface (Fig. 3) it holds that

$$\hat{\boldsymbol{\sigma}}_c = \frac{1}{1-r}\boldsymbol{a} \quad \text{when} \quad f(\hat{\boldsymbol{\sigma}}_c) = F \tag{25}$$

where

$$\boldsymbol{a} \equiv \bar{\boldsymbol{\alpha}} - \hat{\boldsymbol{\alpha}}. \tag{26}$$

By relation (25), expression (24) can be expressed as

$$\operatorname{tr}\left(\frac{\partial f}{\partial \boldsymbol{a}}\dot{\boldsymbol{a}}\right) \leqslant (1-r)^n \dot{F} \quad \text{when} \quad f(\boldsymbol{a}) = (1-r)^n F. \tag{27}$$

Further, noting the relation

$$\frac{\partial f}{\partial \boldsymbol{a}} = \frac{n(1-r)^n F}{\operatorname{tr}(\boldsymbol{n}_c \boldsymbol{a})} \boldsymbol{n}_c \tag{28}$$

where

$$\boldsymbol{n}_c \equiv \frac{\dfrac{\partial f}{\partial \boldsymbol{a}}}{\left\| \dfrac{\partial f}{\partial \boldsymbol{a}} \right\|} = \frac{\dfrac{\partial f}{\partial \hat{\boldsymbol{\sigma}}_c}}{\left\| \dfrac{\partial f}{\partial \hat{\boldsymbol{\sigma}}_c} \right\|} \tag{29}$$

the non-intersection condition (27) is rewritten as

$$\operatorname{tr}\left\{ \boldsymbol{n}_c \left(\dot{\boldsymbol{a}} - \frac{1}{n} \frac{\dot{F}}{F} \boldsymbol{a} \right) \right\} \leqslant 0 \quad \text{when} \quad f(\boldsymbol{a}) = (1-r)^n F. \tag{30}$$

In order to satisfy expression (30), one assumes the following relation, presupposing the convexity of the yield surfaces (Fig. 3):

$$\dot{\boldsymbol{a}} - \frac{1}{n} \frac{\dot{F}}{F} \boldsymbol{a} = \dot{\mu} \boldsymbol{\beta} \tag{31}$$

where $\dot{\mu}$ ($\geqslant 0$) is a proportionality factor and

$$\boldsymbol{\beta} \equiv \boldsymbol{\sigma}_y - \boldsymbol{\sigma} \tag{32}$$

which is also expressed as

$$\boldsymbol{\beta} \equiv \frac{1}{r} \bar{\boldsymbol{\sigma}} - \hat{\boldsymbol{\sigma}} \tag{33}$$

by eqns (15), (16) and (18), where $\hat{\boldsymbol{\sigma}}$ is given by eqn (2), letting $\boldsymbol{\sigma}$ mean a current stress on the subyield surface but not a conjugate stress on the normal-yield surface.

From eqn (31) one has

$$\dot{\boldsymbol{\alpha}} = \dot{\bar{\boldsymbol{\alpha}}} + \frac{1}{n} \frac{\dot{F}}{F} \boldsymbol{a} + \dot{\mu} \boldsymbol{\beta} \tag{34}$$

where $\dot{\mu}$ is given as

$$\dot{\mu} = \frac{\operatorname{tr}\left\{ \bar{\boldsymbol{n}} \left(\dot{\hat{\boldsymbol{\sigma}}} - \frac{1}{n} \frac{\dot{F}}{F} \hat{\boldsymbol{\sigma}} \right) \right\}}{\operatorname{tr}(\bar{\boldsymbol{n}} \boldsymbol{\beta})} \tag{35}$$

by substituting eqn (34) into the consistency condition

$$\operatorname{tr}\left\{ \bar{\boldsymbol{n}} \left(\dot{\bar{\boldsymbol{\sigma}}} - \frac{1}{n} \frac{\dot{F}}{F} \bar{\boldsymbol{\sigma}} \right) \right\} = 0 \tag{36}$$

which is derived by differentiating eqn (14) and substituting the relation

$$\frac{\partial f}{\partial \bar{\sigma}} = \frac{n r^n F}{\operatorname{tr}(\bar{n}\bar{\sigma})} \bar{n}. \tag{37}$$

Finally, one formulates the plastic strain rate so as to satisfy the condition $\dot{\mu} \geqslant 0$. Since the interior of the subyield surface is an elastic region, the associated flow rule holds also for the subyield surface

$$\dot{\varepsilon}^{\mathrm{p}} = \dot{\bar{\lambda}} \bar{n} \tag{38}$$

where $\dot{\bar{\lambda}}$ is a positive proportionality factor.

The inequality $\dot{\mu} \geqslant 0$ results in

$$\operatorname{tr}\left\{\bar{n}\left(\dot{\bar{\sigma}} - \frac{1}{n}\frac{\dot{F}}{F}\bar{\sigma}\right)\right\} \geqslant 0 \tag{39}$$

noting that

$$\operatorname{tr}(\bar{n}\beta) \geqslant 0 \tag{40}$$

in eqn (35).

Substituting eqns (21) and (38) into expression (39) leads to

$$\operatorname{tr}(\bar{n}\dot{\sigma}) - \bar{D}\dot{\bar{\lambda}} \geqslant 0 \tag{41}$$

where

$$\bar{D} \equiv \frac{1}{n}\frac{F'}{F}\bar{\kappa} \operatorname{tr}(\bar{n}\bar{\sigma}) + A \operatorname{tr}^2 \bar{n} + B \operatorname{tr}^2\left(\bar{n}\frac{\bar{\sigma}}{\|\bar{\sigma}\|}\right) \tag{42}$$

$\bar{\kappa}$ is a scalar function of stress, plastic internal state variables and $\bar{n}$ in degree one, which is given by

$$\bar{\kappa} = \dot{K}/\dot{\bar{\lambda}}. \tag{43}$$

Also, $\bar{\kappa}$ is the function that the argument n is replaced by $\bar{n}$ in the function κ.

Now, noting that the plastic strain rate equation must reduce to eqn (10) in the normal-yield state ($\sigma = \sigma_y$), one assumes for $\dot{\bar{\lambda}}$ to be given by

$$\dot{\bar{\lambda}} = \frac{\operatorname{tr}(\bar{n}\dot{\sigma})}{h\bar{D} + H} \tag{44}$$

where h and H are functions of the parameter

$$b = \frac{1}{F^{1/n}} \operatorname{tr}(\bar{n}\beta) \tag{45}$$

satisfy the condition

$$h = 1 \quad \text{and} \quad H = 0 \quad \text{when} \quad b = 0. \tag{46}$$

The parameter b is interpreted as the measure to describe the approaching degree to the normal-yield state (Hashiguchi, 1981).

By letting the left-hand side of expression (41) be denoted by χ and substituting eqn (44) into it, one has

$$\dot{\chi} = \{(h-1)\bar{D}+H\}\dot{\bar{\lambda}} \tag{47}$$

which is not necessarily nonnegative.

In the case of

$$h \neq 1 \quad \text{and} \quad H = 0 \tag{48}$$

one has

$$\dot{\chi} = (h-1)\bar{D}\dot{\bar{\lambda}} \quad \left(= \frac{h-1}{h}\,\mathrm{tr}\,(\bar{n}\dot{\sigma})\right) \tag{49}$$

which also is not necessarily nonnegative.

In the case of

$$h = 1 \quad \text{and} \quad H \geqslant 0 \tag{50}$$

one has

$$\dot{\chi} = H\dot{\bar{\lambda}} \tag{51}$$

which is nonnegative, and therefore expression (39) or (41) is always satisfied.

Eventually, the plastic strain rate is given from eqns (38), (44) and (50) as follows:

$$\dot{\varepsilon}^{p} = \frac{\mathrm{tr}\,(\bar{n}\dot{\sigma})}{\bar{D}+H}\bar{n}. \tag{52}$$

Here, note that in the hardening state satisfying tr $(\bar{n}\dot{\sigma}) > 0$ and $\bar{D} > 0$, the smaller the parameter h the larger the magnitude of plastic strain rate is. From this one knows that the function H for the general material with a hardening behavior should be a monotonically increasing function of h.

The above modification based on the non-intersection condition is required also for the multi surface model formulation (Mroz, 1967).

THREE SURFACE MODEL

The two surface model brings about an abrupt change of the strain rate/stress rate relation when loaded from the stress state within the subyield surface since the interior of this surface is assumed to be a purely elastic domain, so that it cannot describe a smooth elastic–plastic transition. This trend is remarkable especially when a current stress passes through the contact point of the normal-yield and the subyield surfaces. As a more fundamental problem, the loading criterion must include the judgement whether a current stress lies on the subyield surface or not, i.e. yield condition (14) which leads to a disadvantage in a stress/strain calculation, since it assumes an elastic domain.

Now, one extends the foregoing two surface model so as to express the smooth elastic–plastic transition in accordance with the three surface theory (Hashiguchi, 1981) which incorporates a subloading surface (Hashiguchi, 1980) within the subyield surface. The subloading surface is the surface that is geometrically similar to the subyield surface with

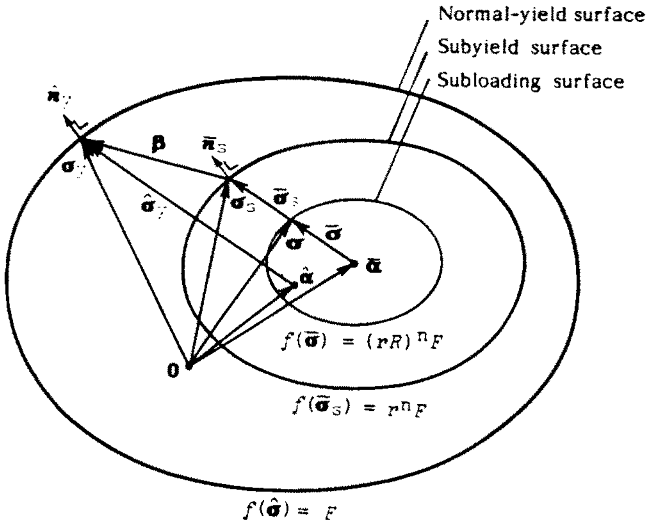

Fig. 4. Normal-yield, subyield and subloading surfaces.

respect to $\bar{\alpha}$ (Fig. 4) and passes always through a current stress point not only in a loading state but also in an unloading state, and thus the loading criterion need not include a judgement whether a stress lies on a subyield surface or not. In the subloading surface concept the following parameter is introduced:

$$R = \frac{1}{r}\left\{\frac{f(\bar{\sigma})}{F}\right\}^{1/n} \tag{53}$$

which designates the ratio of the size of the loading surface to that of the subyield surface. Hereinafter let the conjugate point on the subyield surface be denoted by σ_s and then it holds that

$$\bar{\sigma}_s = \bar{\sigma}/R \tag{54}$$

$$\bar{n}_s = \bar{n} \tag{55}$$

where

$$\bar{\sigma}_s \equiv \sigma_s - \bar{\alpha} \tag{56}$$

$$\bar{n}_s \equiv \frac{\partial f}{\partial \bar{\sigma}_s}\Bigg/\left\|\frac{\partial f}{\partial \bar{\sigma}_s}\right\|. \tag{57}$$

Differentiation of eqn (53) leads to

$$\mathrm{tr}\,(\bar{n}\dot{\bar{\alpha}}) = \mathrm{tr}\left[\bar{n}\left\{\dot{\sigma} - \left(\frac{1}{n}\frac{\dot{F}}{F} + \frac{\dot{R}}{R}\right)\bar{\sigma}\right\}\right]. \tag{58}$$

From eqn (36) and rewriting σ as σ_s, noting eqn (55), one has

$$\mathrm{tr}\,(\bar{n}\dot{\sigma}_s) = \mathrm{tr}\left\{\bar{n}\left(\dot{\bar{\alpha}} + \frac{1}{n}\frac{\dot{F}}{F}\bar{\sigma}_s\right)\right\}. \tag{59}$$

Substituting eqn (58) into eqn (59), one obtains

$$\operatorname{tr}(\bar{\boldsymbol{n}}\dot{\boldsymbol{\sigma}}_s) = \operatorname{tr}\left[\bar{\boldsymbol{n}}\left\{\dot{\boldsymbol{\sigma}} + \frac{1}{n}\frac{\dot{F}}{F}(\boldsymbol{\sigma}_s - \boldsymbol{\sigma}) - \frac{\dot{R}}{R}\bar{\boldsymbol{\sigma}}\right\}\right]. \tag{60}$$

Here, one assumes that $\dot{R}/R$ is nonnegative in the loading state and is given by

$$\dot{R}/R = U\|\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}\| \tag{61}$$

where U ($\geqslant 0$) is a monotonically decreasing function of R, satisfying the condition

$$U = 0 \quad \text{when} \quad R = 1. \tag{62}$$

Further, assume that the associated flow rule holds also for the loading surface. Then, eqn (61) is rewritten as

$$\dot{R}/R = U \operatorname{tr}(\bar{\boldsymbol{n}}\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}) \tag{63}$$

or

$$\dot{R}/R = U\dot{\bar{\lambda}}. \tag{64}$$

Substituting eqns (60) and (61) into eqn (34) with eqn (35) and eqn (52) with eqn (42) with regarding $\boldsymbol{\sigma}$ in equations of the two surface model to be $\boldsymbol{\sigma}_s$, one has

$$\dot{\bar{\boldsymbol{\alpha}}} = \dot{\boldsymbol{\alpha}} + \frac{1}{n}\frac{\dot{F}}{F}\boldsymbol{a} + \dot{\bar{\mu}}\boldsymbol{\beta} \tag{65}$$

$$\dot{\bar{\mu}} = \frac{\operatorname{tr}\left\{\bar{\boldsymbol{n}}\left(\dot{\boldsymbol{\sigma}} - \frac{1}{n}\frac{\dot{F}}{F}\hat{\boldsymbol{\sigma}} - U\|\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}\|\bar{\boldsymbol{\sigma}}\right)\right\}}{\operatorname{tr}(\bar{\boldsymbol{n}}\boldsymbol{\beta})} \tag{66}$$

$$\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}} = \frac{\operatorname{tr}(\bar{\boldsymbol{n}}\dot{\boldsymbol{\sigma}})}{\bar{D} + H + U\operatorname{tr}(\bar{\boldsymbol{n}}\bar{\boldsymbol{\sigma}})}\bar{\boldsymbol{n}} \tag{67}$$

$\boldsymbol{\sigma}$ in $\bar{D}$ given by eqn (42) being a current stress. $\dot{\boldsymbol{\alpha}}$ and b are given by the same equations as eqns (21) and (45), respectively.

The two surface model predicts a discontinuous stress rate/strain rate relation and is incapable of describing a plastic deformation due to the stress change within the subyield surface the interior of which is assumed to be a purely elastic domain. These shortcomings are excluded in the three surface model. On the other hand, the three surface model predicts an excessive mechanical ratchetting effect, since the center of similarity of the subyield and the normal-yield surfaces is fixed in the center of the subyield surface while an unloading (elastic deformation) proceeds until the stress returns the center of similarity. This shortcoming would be modified by letting the center of similarity translate within the subyield surface.

A LOADING CRITERION

The plastic strain rate obeying the associated flow rule is generally given by

$$\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}} = \dot{\lambda}\boldsymbol{n} \tag{68}$$

where

$$\dot{\lambda} \equiv \frac{\mathrm{tr}\,(\boldsymbol{n}\dot{\boldsymbol{\sigma}})}{L} \tag{69}$$

as seen in eqns (10), (52) and (67). In eqns (68) and (69), $\dot{\lambda}$ is a positive proportionality factor, $\boldsymbol{n}$ a unit outer normal of the loading surface and L a function of stress and some plastic internal state variables.

By substituting

$$\dot{\boldsymbol{\sigma}} = \boldsymbol{E}\dot{\boldsymbol{\varepsilon}}^{e} \tag{70}$$

$$\dot{\boldsymbol{\varepsilon}}^{e} = \dot{\boldsymbol{\varepsilon}} - \dot{\boldsymbol{\varepsilon}}^{p} \tag{71}$$

and eqn (68) into eqn (69), one has

$$\dot{\lambda} = \frac{\mathrm{tr}\,\{\boldsymbol{n}\boldsymbol{E}(\dot{\boldsymbol{\varepsilon}} - \dot{\lambda}\boldsymbol{n})\}}{L} \tag{72}$$

where $\dot{\boldsymbol{\varepsilon}}$ is a strain rate, $\dot{\boldsymbol{\varepsilon}}^{e}$ an elastic strain rate and the fourth-order tensor $\boldsymbol{E}$ an elastic modulus. While from eqn (72) one obtains the expression of $\dot{\lambda}$ by the strain rate instead of the stress rate, let it be denoted by $\dot{\Lambda}$

$$\dot{\Lambda} \equiv \frac{\mathrm{tr}\,(\boldsymbol{n}\boldsymbol{E}\dot{\boldsymbol{\varepsilon}})}{L + \mathrm{tr}\,(\boldsymbol{n}\boldsymbol{E}\boldsymbol{n})}. \tag{73}$$

It holds that

$$\mathrm{tr}\,(\boldsymbol{n}\dot{\boldsymbol{\sigma}}) \leqslant 0 \tag{74}$$

in the unloading state, while expression (74) holds also in the loading state with a softening. Since it holds that $\dot{\boldsymbol{\sigma}} = \boldsymbol{E}\dot{\boldsymbol{\varepsilon}}$ in the unloading, expression (74) is rewritten as

$$\mathrm{tr}\,(\boldsymbol{n}\boldsymbol{E}\dot{\boldsymbol{\varepsilon}}) \leqslant 0 \tag{75}$$

in the unloading state.

If $L + \mathrm{tr}\,(\boldsymbol{n}\boldsymbol{E}\boldsymbol{n}) < 0$, a plastic deformation cannot occur for $\mathrm{tr}\,(\boldsymbol{n}\boldsymbol{E}\dot{\boldsymbol{\varepsilon}}) > 0$ by the collateral condition $\dot{\Lambda} > 0$ for $\dot{\boldsymbol{\varepsilon}}^{p} \neq \boldsymbol{0}$. Further, if $L + \mathrm{tr}\,(\boldsymbol{n}\boldsymbol{E}\boldsymbol{n}) = 0$, a plastic deformation can occur only for $\mathrm{tr}\,(\boldsymbol{n}\boldsymbol{E}\dot{\boldsymbol{\varepsilon}}) = 0$. These facts and expression (75) lead to the shortcoming that any deformation cannot occur for $\mathrm{tr}\,(\boldsymbol{n}\boldsymbol{E}\dot{\boldsymbol{\varepsilon}}) > 0$, while real materials can undergo a strain rate of any direction (this point is fundamentally different from a stress rate). On the other hand, if $L + \mathrm{tr}\,(\boldsymbol{n}\boldsymbol{E}\boldsymbol{n}) > 0$, a plastic deformation proceeds for $\mathrm{tr}\,(\boldsymbol{n}\boldsymbol{E}\dot{\boldsymbol{\varepsilon}}) > 0$, while an elastic deformation proceeds for $\mathrm{tr}\,(\boldsymbol{n}\boldsymbol{E}\dot{\boldsymbol{\varepsilon}}) \leqslant 0$. Eventually, in order to describe a strain rate of any direction, it must hold that

$$L + \mathrm{tr}\,(\boldsymbol{n}\boldsymbol{E}\boldsymbol{n}) > 0 \tag{76}$$

while L can take both positive and negative values but $\mathrm{tr}\,(\boldsymbol{n}\boldsymbol{E}\boldsymbol{n}) > 0$ since $\mathrm{tr}\,(\boldsymbol{n}\boldsymbol{E}\boldsymbol{n})$ is of the quadratic form and $\boldsymbol{E}$ is a positive definite as known from $\mathrm{tr}\,(\dot{\boldsymbol{\varepsilon}}^{e}\boldsymbol{E}\dot{\boldsymbol{\varepsilon}}^{e}) = \mathrm{tr}\,(\dot{\boldsymbol{\sigma}}\dot{\boldsymbol{\varepsilon}}^{e}) > 0$ ($\dot{\boldsymbol{\varepsilon}}^{e} \neq \boldsymbol{0}$). A loading criterion should be given as

$$\begin{aligned} \dot{\boldsymbol{\varepsilon}}^{p} &\neq \boldsymbol{0} \quad \text{for} \quad \mathrm{tr}\,(\boldsymbol{n}\boldsymbol{E}\dot{\boldsymbol{\varepsilon}}) > 0 \\ \dot{\boldsymbol{\varepsilon}}^{p} &= \boldsymbol{0} \quad \text{for} \quad \mathrm{tr}\,(\boldsymbol{n}\boldsymbol{E}\dot{\boldsymbol{\varepsilon}}) \leqslant 0 \end{aligned} \tag{77}$$

while constitutive equations with a purely elastic domain require a further judgement whether a yield condition (eqn (1) for the classical theory and eqn (14) for the two surface model) is satisfied or not. Equations (77) were shown by Hill (1958) presupposing $L > 0$, i.e. a hardening material and assumed *a priori* by Mroz and Zienkiewicz (1984) in a different approach, i.e. a strain space formulation of plastic constitutive equations.

On the other hand, consider $\dot{\lambda}$ in eqn (69) expressed by a stress rate. The loading in the state $L < 0$ brings about a softening in which tr $(\boldsymbol{n}\dot{\boldsymbol{\sigma}}) < 0$ and $L < 0$ leads to $\dot{\lambda} > 0$. However, the unloading tr $(\boldsymbol{n}\dot{\boldsymbol{\sigma}}) < 0$ in the state $L < 0$ also leads to $\dot{\lambda} > 0$. Thus, $\dot{\lambda} > 0$ is a necessary condition but not a sufficient condition for a loading state. Eventually, $\dot{\lambda}$ cannot be adopted as a quantity to define a loading criterion for materials exhibiting not only a hardening but also a softening behavior.

COMMENTS ON PAST MODELS AND FORMULATIONS

Various models assuming plural surfaces have been proposed in the past: the two, the multi and the infinite surface models. In what follows, some comments on these past models and formulations are made comparing with the two and the three surface models formulated in this paper.

Translation rule of subyield surface

In the multi surface model, so-called a "model of a field of hardening moduli" proposed by Mroz (1967), the translation rule of the active subyield surface was assumed *a priori* as follows:

$$\dot{\boldsymbol{\alpha}} = \dot{\mu}_1 \boldsymbol{\beta} \tag{78}$$

which was used by Mroz (1969), Prevost (1977, 1978, 1982), Faruque (1985) and Gilat (1985) for the multi surface model formulation and by Krieg (1975), Lamba and Sidebottom (1978), Prevost (1982), Chaboche and Rousselier (1983), Bruhns (1984), Mcdowell (1985a, b), Shaw and Kyriakides (1985), Chaboche (1986) and Ohno and Kachi (1986) for the two surface model formulation. Equation (78) coincides with eqn (34) in the case of the non-hardening normal-yield surface, i.e. $\dot{F} = 0$ and $\dot{\boldsymbol{\alpha}} = \mathbf{0}$.

Also for the multi surface model formulation, Mroz *et al.* (1978) adopted, however, the different equation

$$-\dot{\boldsymbol{\beta}} = \dot{\mu}_2 \boldsymbol{\beta} \tag{79}$$

which was used by Mroz *et al.* (1979) and Hashiguchi (1981) for the two surface model formulation.

Further, Dafalias and Popov (1975, 1976, 1977) incorporated the equation

$$\dot{\mathrm{a}} = \dot{\mu}_3 \boldsymbol{\beta} \tag{80}$$

for the two surface model formulation. In eqns (78)–(80), $\dot{\mu}_1$, $\dot{\mu}_2$ and $\dot{\mu}_3$ are positive proportionality factors which can be expressed explicitly by substituting these equations into the consistency condition (36), whereas they do not examine the positivity of $\dot{\mu}_1$, $\dot{\mu}_2$ and $\dot{\mu}_3$ in their formulations of plastic strain rate equations.

Equations (68)–(70) do not satisfy generally the non-intersection condition (30).

Measure of approaching degree to the normal-yield state

In the formulation of the two surface model, Krieg (1975), Dafalias and Popov (1975) and Mroz *et al.* (1979) assumed the parameter

$$b' \equiv \|\boldsymbol{\beta}\|/F^{1/n} \tag{81}$$

as a measure to describe the approaching degree to the normal-yield state.

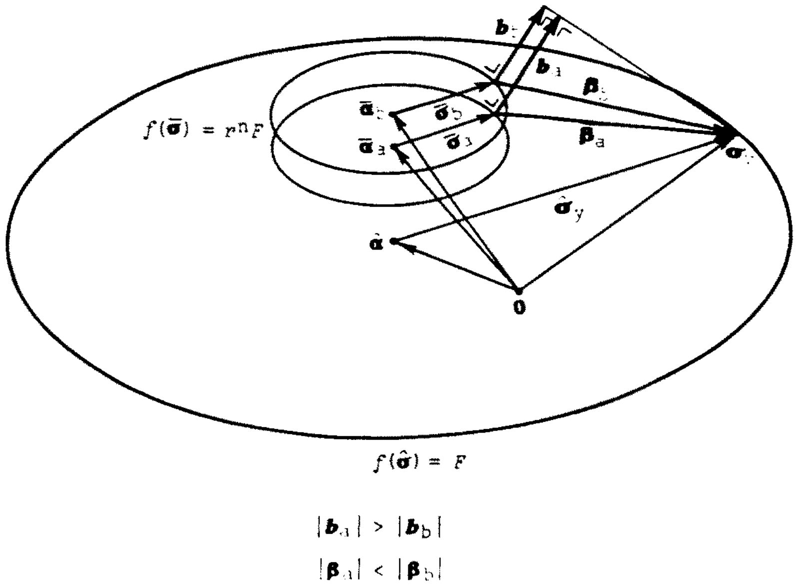

Fig. 5. A measure of approaching degree to the normal-yield state.

As the current stress approaches the normal-yield state, b becomes smaller. On the other hand, there exists the case that b' becomes larger inversely in that approach as illustrated in Fig. 5 in which $\boldsymbol{b} = \mathrm{tr}\,(\bar{\boldsymbol{n}}\boldsymbol{\beta})\bar{\boldsymbol{n}} = bF^{1/n}\bar{\boldsymbol{n}}$.

While b would be a reasonable measure of approach to the normal-yield state, b' is unacceptable as that measure.

The idea of vanishing elastic domain

Dafalias and Popov (1977), Mroz *et al.* (1979, 1981) and Mroz (1980) insisted the validity of the idea of a vanishing elastic region as the limiting case of the two surface model, in which the subyield surface diminishes to a point. Further, they assumed that a conjugate point on the normal-yield surface lies on the extension of the stress rate vector stemming from the current stress point or the center of the normal-yield surface. This idea may be expected to be capable of describing the dependence of the direction of the plastic strain rate on the stress rate, that is, the "pseudo-corner effect" and also the smooth elastic–plastic transition.

For example, consider the softening state as shown in Fig. 6. There occurs a fatal

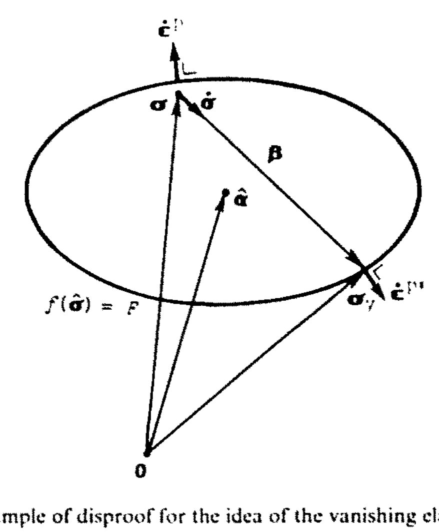

Fig. 6. An example of disproof for the idea of the vanishing elastic domain.

contradiction that the conjugate point lies on the opposite side of the normal-yield surface the outer-normal direction of which differs entirely from the actual direction of the plastic strain rate $\dot{\varepsilon}^p$, predicting the plastic strain rate $\dot{\varepsilon}^p$ with an extremely large value of b (or b').

It would be adequate to assume a small subyield surface in order to express some dependence of the direction of plastic strain rate on the stress rate. It would be, however, physically unreasonable that the direction of the plastic strain rate depends mainly on the stress rate but not on the stress state as is predicted by the idea of a vanishing elastic region. This idea describes the stress rate dependence stronger than the corner effect of the yield surface, since it makes the yield surface shrink to a point. Eventually, this idea is not acceptable physically and mathematically at present.

Expansion rule of loading surface

The author (Hashiguchi, 1980) assumed the equation

$$\dot{R}/R = U\dot{F} \tag{82}$$

in the previous three surface model. According to this equation, the current stress cannot approach to the subyield surface but recedes from it in the non-hardening state $\dot{F} \leqslant 0$. On the other hand, eqn (61) would be applicable to the generalized material with not only hardening but also softening behavior.

The multi and the infinite surface models

In the multi surface model which was proposed by Mroz (1967) and Iwan (1967) independently, the current stress transfers steadily to larger active subyield surfaces. This model needs to memorize all of the assumed surfaces. Further, a reloading after a partial reverse loading produces an abrupt change of the stress rate/strain rate relation when the stress passes through the stress reversed point because the stress transfers abruptly to the large subyield surface which was an active subyield surface just before the reverse loading took place.

Mroz *et al.* (1981), Mroz and Norris (1982) and Mroz and Pietruszczak (1983) refined the multi surface model mathematically by imaging a field of an infinite number of subyield surfaces the sizes of which range from a point to the normal-yield surface. A salient feature of this formulation is the hypothesis of "stress reversal surface". This is the surface which was an active subyield surface just before a reverse loading took place. Then, a new active subyield surface expands contacting with the stress reversal surface at the stress reversed point. This model, named an "infinite surface model", may be regarded also as a reasonable formalization of the loading surface concept introduced by Greenstreet and Phillips (1973) for artificial graphite. It is a notable improvement of the multi surface model because it needs to memorize only three kinds of surface: an active subyield surface on which a current stress exists, a normal-yield (bounding or limiting) surface and stress reversal surfaces. In the case of the cyclic loading with decreasing amplitudes, however, many stress reversal surfaces should be memorized, on which stress reversal events took place before then. Further, the abrupt change of the hardening modulus in a reloading after a partial reverse loading cannot be avoided also by that model.

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